

A QUADRATIC SEPARATION INEQUALITY FOR SPANNING TREES

GEFEI CAI

ABSTRACT. We prove a quadratic separation inequality for weighted spanning trees on finite undirected networks and give two applications. First, for every finite n -vertex transitive network, the expected distance in the uniform spanning tree (UST) between two independent uniform vertices is at least $\sqrt{n}-1$, resolving a conjecture in [Benjamini–Kozma, *Comm. Math. Phys.* **259** (2005), 257–286]. Second, we prove that the growth exponent of three-dimensional loop-erased random walk (LERW) is at least $3/2$.

1. INTRODUCTION

Let $G = (V, E)$ be a finite connected undirected graph, with positive edge conductances $(c_e)_{e \in E}$ and a distinguished root $r \in V$. Parallel edges are allowed; loops may be discarded. A weighted spanning tree T has distribution

$$(1.1) \quad \mathbb{P}(T = t) = Z_G^{-1} w(t), \quad w(t) = \prod_{e \in t} c_e, \quad Z_G = \sum_{t \text{ spanning tree}} w(t).$$

Write $\gamma_x = [x, r]_T$ for its root path, including both endpoints, and $\text{LCA}_T(x, y)$ for the first common vertex of γ_x and γ_y when followed towards r . Tree distances count edges, whereas the cardinality of a path counts vertices. We call a network *transitive* if its conductance-preserving automorphisms act transitively on its vertices.

For $B \subseteq V \setminus \{r\}$, define

$$(1.2) \quad H_B(x) = \mathbb{P}(\gamma_x \cap B \neq \emptyset), \quad a_B(x, y) = \mathbb{P}(\text{LCA}_T(x, y) \in B).$$

Let $U_B = V \setminus (B \cup \{r\})$. For distinct $x, y \in U_B$, let $J_B(x, y)$ be the probability that x, y, r belong to three different components of the induced forest $T \setminus B$, and put $J_B(x, x) = 0$. The vector H_B in the next display is restricted to U_B .

Theorem 1.1 (Quadratic separation inequality). *For every G, r, B as above,*

$$(1.3) \quad H_B H_B^\top - J_B \succeq 0.$$

Consequently, for every nonnegative $f : V \rightarrow \mathbb{R}$,

$$(1.4) \quad \sum_{x, y \in V} f(x) f(y) a_B(x, y) \leq \left(\sum_{x \in V} f(x) H_B(x) \right)^2.$$

For all $x, y \in V$, one also has

$$(1.5) \quad a_B(x, y) \leq 2H_B(x)H_B(y).$$

Here $M \succeq 0$ means that the real symmetric matrix M is positive semidefinite.

If f is a probability distribution, the left side of (1.4) is a two-branch merging probability with independent, identically distributed starting points. The constant 1 in (1.4) cannot be decreased uniformly. Indeed, take the deterministic tree with edges $\{r, b\}$ and $\{b, x_i\}$, $1 \leq i \leq k$, put $B = \{b\}$, and let f be uniform on the k leaves; the two sides are $1 - 1/k$ and 1.

Theorem 1.1 has the following two main applications.

Theorem 1.2. *Let G be a finite connected transitive network with n vertices. Let T have law (1.1), and let X, Y be independent uniform vertices, independent of T . Write $\gamma = [X, Y]_T$. For every deterministic $B \subseteq V$,*

$$(1.6) \quad \mathbb{P}(\gamma \cap B \neq \emptyset) \geq \sqrt{\frac{|B|}{n}}.$$

For a deterministic partition $\mathcal{P} = \{B_1, \dots, B_k\}$ of V into nonempty sets, let $N_{\mathcal{P}}$ count the blocks visited by γ . Then

$$(1.7) \quad \mathbb{E}N_{\mathcal{P}} \geq \sum_{i=1}^k \sqrt{\frac{|B_i|}{n}}.$$

In particular, let $d_T(X, Y)$ be the number of edges in the unique path from X to Y in T . Then

$$(1.8) \quad \mathbb{E}d_T(X, Y) \geq \sqrt{n} - 1.$$

Benjamini and Kozma [7, Introduction] asked whether the expected UST distance between two independent uniform vertices of every finite vertex-transitive graph admits a universal lower bound of order $\sqrt{n}$. This has been proved under high-dimensional or model-specific assumptions; see e.g. [19, 20, 18]. We also refer to [9, 1, 4, 5] for related results on UST distances, diameters, and scaling limits.

Let S be a simple random walk on $\mathbb{Z}^3$, started at 0. For the Euclidean ball $B_R = \{x \in \mathbb{Z}^3 : |x| < R\}$, set

$$(1.9) \quad \tau_R = \inf\{j \geq 0 : S_j \notin B_R\}, \quad M_R = |\text{LE}(S[0, \tau_R])|, \quad \beta = \lim_{R \rightarrow \infty} \frac{\log \mathbb{E}M_R}{\log R}.$$

The limit exists by Shiraishi [21]. Let $\eta = \text{LE}(S[0, \infty))$ be the infinite loop-erasure, defined by successive last visits.

Theorem 1.3. *There is $c > 0$ such that, for every $R \geq 1$,*

$$(1.10) \quad \mathbb{E}|\eta \cap B_R| \geq cR^{3/2}.$$

Consequently, we have $\beta \geq 3/2$.

Note that β also equals the Hausdorff dimension of the scaling limit of the three-dimensional LERW [22]. The classical bound $\beta > 1$ follows by comparison with random-walk intersection exponents; see Lawler [10, Introduction] and the references therein. Recently, Liu [14] made this approach quantitative and improved it to $\beta > 1.0001$. Theorem 1.3 also implies that two independent infinite LERWs on $\mathbb{Z}^3$ intersect infinitely often a.s.; see Theorem 5.1.

Organization of the paper. Section 2 proves Theorem 1.1 using matroid log-concavity results and closure properties from Anari–Liu–Oveis Gharan–Vinzant [2] and Brändén–Huh [8]. Sections 3 and 4 then prove Theorems 1.2 and 1.3, respectively. Section 5 provides three further consequences of Theorem 1.1, including that the three-dimensional long-range LERW in the non-mean-field regime has lower growth exponent at least $3/2$ as well (Proposition 5.3).

Statement on the use of AI. The original motivation for this work was to prove Theorem 1.3. The author reduced this problem to the sufficient condition (1.5), a general inequality for finite graphs. With the assistance of AI, the author subsequently obtained the proof of Theorem 1.1 and found that the theorem also yields Theorem 1.2 as a corollary. The author has verified all the proofs and takes full responsibility for the paper.

2. THE QUADRATIC SEPARATION INEQUALITY

A polynomial with nonnegative coefficients is *completely log-concave* if every iterated directional derivative (including order zero) in nonnegative directions is either zero or has concave logarithm on the positive orthant. We use the following input from the theory of log-concave matroid polynomials [2, 8].

Theorem 2.1. *Let M be a matroid on a finite ground set F , with independent sets $\mathcal{I}(M)$. Then*

$$(2.1) \quad \sum_{I \in \mathcal{I}(M)} s^{|F|-|I|} \prod_{e \in I} z_e \quad \text{and} \quad \sum_{\substack{A \subseteq F \\ A \text{ a basis of } M}} \prod_{e \in A} z_e$$

are homogeneous completely log-concave polynomials. Moreover, the class of homogeneous completely log-concave polynomials is closed under products, partial derivatives, and nonnegative linear substitutions; the resulting polynomial is allowed to be identically zero.

The first polynomial in (2.1) is written without factorial normalization. Its complete log-concavity is proved in [2, Theorem 4.1]; both polynomials are covered by [8, Proposition 4.25 and Theorem 2.30]. The closure properties follow from [8, Theorem 2.10 and Corollary 2.32].

For clarity, a matroid is a nonempty hereditary family of subsets satisfying the augmentation axiom; its maximal independent sets are its bases. We use only graphic matroids, whose independent sets are forests, free matroids, in which every subset is independent, and dual matroids, whose bases are complements of bases. Direct sums impose the two independence conditions on disjoint ground sets.

For a realization of the weighted spanning tree T , write $\mathcal{C}_B(T) = \text{Comp}(T \setminus B)$ and let $C_r(T)$ be the component containing r . For variables $z = (z_u)_{u \in U_B}$, define

$$(2.2) \quad Q_B(z) = \mathbb{E} \prod_{C \in \mathcal{C}_B(T) \setminus \{C_r(T)\}} \left(1 + \sum_{u \in C} z_u \right).$$

An empty product is 1.

Proposition 2.2. *The polynomial Q_B is completely log-concave.*

Proof. Put $U = V \setminus B$, $k = |V| - 1$, and $E_U = E(G[U])$. Add a new vertex $*$ and the star edges $M = \{s_u = \{*, u\} : u \in U\}$. On $E \sqcup M$, take the direct sum N of the graphic matroid of $G[U] + M$ and the free matroid on $E \setminus E_U$. Set $N = |E| + |U|$.

Introduce variables t_e for $e \in E$, variables z_u for $u \in U$, and a homogenizing variable s . The polynomial

$$(2.3) \quad I(s, t, z) = \sum_{J \in \mathcal{I}(N)} s^{N-|J|} \prod_{e \in J \cap E} (c_e t_e) \prod_{s_u \in J \cap M} z_u$$

is homogeneous completely log-concave by Theorem 2.1 and positive rescaling. So is the dual graphic basis polynomial

$$(2.4) \quad B^*(t) = \sum_{T \text{ spanning tree of } G} \prod_{e \in E \setminus T} t_e.$$

Extract the square-free coefficient in the original edge variables:

$$(2.5) \quad F(s, z) = \left[\prod_{e \in E} t_e \right] (I(s, t, z) B^*(t)) = \left(\prod_{e \in E} \partial_{t_e} \right) (I B^*) \Big|_{t=0}.$$

The closure properties above show that F is homogeneous completely log-concave, of degree $D = N - k$. For a term indexed by T in B^* , the surviving original edges in I are exactly T . A star-edge set can be added to $T \cap E_U$ without a cycle precisely when it selects at most one vertex from each component of $T \setminus B$. Consequently,

$$(2.6) \quad F(s, z) = \sum_T w(T) s^{D-|\mathcal{C}_B(T)|} \prod_{C \in \mathcal{C}_B(T)} \left(s + \sum_{u \in C} z_u \right).$$

All displayed exponents are nonnegative, since $D \geq |U| \geq |\mathcal{C}_B(T)|$.

Differentiate in the temporary variable z_r and set $s = 1$. Only the factor belonging to $C_r(T)$ is differentiated, so

$$(2.7) \quad Q_B(z) = Z_G^{-1} \partial_{z_r} F(s, z) \Big|_{s=1}.$$

Partial differentiation preserves homogeneous complete log-concavity; restriction to $s = 1$ preserves log-concavity of every derivative in the remaining variables. This proves the proposition. $\square$

Proof of Theorem 1.1. By (2.2),

$$(2.8) \quad Q_B(0) = 1, \quad \partial_x Q_B(0) = H_B(x), \quad \partial_x \partial_y Q_B(0) = J_B(x, y) \quad (x, y \in U_B).$$

The diagonal second derivatives vanish because a variable occurs in at most one factor in each product. Taking the Hessian of $\log Q_B$ at the origin yields (1.3). This boundary evaluation is legitimate by continuity from the positive orthant, since $Q_B(0) = 1$.

For $x, y \in U_B$, the event $\{\text{LCA}_T(x, y) \in B\}$ implies that x, y, r lie in three different components of $T \setminus B$. Thus $a_B(x, y) \leq J_B(x, y)$, and (1.3) implies (1.4) for f supported on U_B . The two-by-two principal minors of (1.3) give

$$(2.9) \quad (H_B(x)H_B(y) - J_B(x, y))^2 \leq H_B(x)^2 H_B(y)^2,$$

and hence $J_B(x, y) \leq 2H_B(x)H_B(y)$.

It remains to allow sources in B or at r . Terms involving r vanish, while $H_B(x) = 1$ for $x \in B$ and $a_B(x, y) \leq H_B(y)$ whenever $x \in B$. Put $a = \sum_{x \in B} f(x)$ and $b = \sum_{x \in U_B} f(x)H_B(x)$. The contributions with zero, one, or two sources in B are bounded by b^2 , $2ab$, and a^2 , respectively. This proves (1.4) for arbitrary nonnegative f , and also handles the remaining cases of (1.5). $\square$

3. PROOF OF THEOREM 1.2

We prove Theorem 1.2 by applying the separation inequality before averaging the root. For a finite network G and $m > 0$, adjoin a cemetery vertex ∂ and an edge of conductance m from each $v \in V$ to ∂ . Deleting ∂ from a weighted spanning tree of the augmented network yields a rooted forest $\mathfrak{F}_m$, called the *massive spanning forest* with constant killing rate m . Each component is rooted at its unique vertex previously adjacent to ∂ . Its component containing x is $C_m(x)$, and γ_x^m is the path from x to ∂ , with ∂ omitted. Wilson's algorithm identifies this path with the loop-erasure of the continuous-time walk with jump rates $c_{xy} = \sum_{e=\{x,y\}} c_e$, stopped at an independent exponential time of rate m [16, Chapter 4].

Proof of Theorem 1.2. Fix $B \subseteq V$, and let U, W be independent uniform vertices, independent of the augmented tree. Apply (1.4) with root ∂ and $f = \mathbf{1}_V/n$:

$$(3.1) \quad \mathbb{P}_m(\text{LCA}(U, W) \in B) \leq \mathbb{P}_m(\gamma_U^m \cap B \neq \emptyset)^2.$$

An augmented tree with j cemetery edges has weight proportional to m^j . As $m \downarrow 0$, its law therefore converges to a weighted spanning tree T of G with an independent uniform vertex R joined to ∂ . The joint law of (T, U, W, R) is invariant under the transitive automorphism group, so $\text{LCA}_R(U, W)$ is uniform on V . All quantities concern a fixed finite graph; taking the limit in (3.1) gives

$$\frac{|B|}{n} \leq \mathbb{P}([U, R]_T \cap B \neq \emptyset)^2,$$

which proves (1.6). For a partition $\mathcal{P}$, sum the indicators that the path meets each block to obtain (1.7). The partition into singletons gives $\mathbb{E}|\gamma| \geq \sqrt{n}$. Since $|\gamma| = d_T(X, Y) + 1$, this proves (1.8). $\square$

4. PROOF OF THEOREM 1.3

We prove Theorem 1.3 directly from local connectivity estimates in massive spanning forests. The construction of $\mathfrak{F}_m$ extends to every countable locally finite network with symmetric jump rates satisfying $\sum_y c_{xy} = 1$. Indeed, for finitely many starting points, the walks in Wilson's algorithm have finite lifetimes almost surely. Exhaustion by finite wired sets, retaining the killing rate m , therefore gives convergence of their entire branches and

common ancestors. In particular, Theorem 1.1 applies to finite source and target sets in $\mathfrak{F}_m$. We set $\text{LCA}(x, y) = \partial$ when x, y belong to distinct spatial components.

Proof of Theorem 1.3. We use rate-one continuous time, which leaves loop-erasure unchanged. We first obtain a local connectivity estimate at killing rate R^{-2} , then decrease the killing rate while preserving that connection event.

Local connectivity estimate. We record the intersection calculation for a general symmetric transitive rate-one network, since it will also be used in Section 5.2. Let $p_t(x, y)$ be its transition kernel, fix an origin o , and put

$$(4.1) \quad g_m(x, y) = \int_0^\infty e^{-mt} p_t(x, y) dt, \quad \mathcal{B}(m) = \sum_z g_m(o, z)^2.$$

Symmetry and the semigroup identity give

$$(4.2) \quad \sum_z g_m(x, z) = m^{-1}, \quad \mathcal{B}(m) = \int_0^\infty t e^{-mt} p_t(o, o) dt.$$

For independent walks X, Y started at x, y , with independent rate- m exponential lifetimes ζ_X, ζ_Y , define

$$(4.3) \quad I_{x,y} = \int_0^{\zeta_X} \int_0^{\zeta_Y} \mathbf{1}_{\{X_s=Y_t\}} dt ds.$$

Writing $g = g_m$, the Markov property and the four orderings of two times on each walk yield

$$(4.4) \quad \mathbb{E} I_{x,y} = \sum_u g(x, u) g(y, u),$$

$$(4.5) \quad \mathbb{E} I_{x,y}^2 = 2 \sum_{u,v} g(x, u) g(y, u) g(u, v)^2 + 2 \sum_{u,v} g(x, u) g(y, v) g(u, v)^2.$$

For instance, the ordering $s < s', t' < t$ contributes $g(x, u) g(u, v) g(y, v) g(v, u)$. The time diagonals have Lebesgue measure zero, and Tonelli's theorem justifies the calculation. Each sum in (4.5), before its factor 2, is at most $\mathcal{B}(m)^2$: for the first use Cauchy–Schwarz, and for the second use $ab \leq (a^2 + b^2)/2$ and the row and column sums $\mathcal{B}(m)$ of $g(u, v)^2$. Thus

$$(4.6) \quad \mathbb{E} I_{x,y}^2 \leq 4\mathcal{B}(m)^2.$$

The one-sided loop-erasure inequality of Lyons–Peres–Schramm [17, Lemma 1.2 and Remark 1.3], followed by Wilson's algorithm, gives

$$(4.7) \quad \mathbb{P}_m(y \in C_m(x)) = \mathbb{P}(\text{LE}(X[0, \zeta_X]) \cap Y[0, \zeta_Y] \neq \emptyset) \geq 2^{-8} \mathbb{P}(I_{x,y} > 0).$$

Only spatial vertices are counted in this intersection.

Now specialize to $\mathbb{Z}^3$, set $\mu = R^{-2}$, and take $x, y \in B_R$. The local central limit estimates imply

$$(4.8) \quad \inf_{x,u \in B_R} g_\mu(x, u) \geq cR^{-1}, \quad \mathcal{B}(\mu) \leq CR.$$

The first estimate follows by integrating over $t \in [R^2, 2R^2]$; the second follows from $p_t(0, 0) \leq C(1+t)^{-3/2}$ and (4.2) [12, Theorem 2.1.3]. Combining (4.4) and (4.6) gives $\mathbb{E} I_{x,y} \geq cR$ and $\mathbb{E} I_{x,y}^2 \leq CR^2$. Hence $\mathbb{P}(I_{x,y} > 0) \geq c$, and (4.7) gives a connection probability at least $a_0 > 0$, uniformly in $x, y \in B_R$.

Write $x \longleftrightarrow_D y$ when the spatial forest path from x to y is contained in D . Doob's inequality, applied up to the independent killing time of mean R^2 , gives

$$\sup_{x \in B_R} \mathbb{P}_x \left(\sup_{t < \zeta_X} |X_t| \geq KR \right) \leq \frac{C}{(K-1)^2}.$$

If both raw walks stay in B_{KR} and the second hits the loop-erasure of the first, Wilson's connecting path stays in that ball. Choose a fixed $K > 1$ so that the two confinement errors sum to at most $a_0/2$. Then, with $a = a_0/2$,

$$(4.9) \quad \inf_{x, y \in B_R} \mathbb{P}_\mu(x \longleftrightarrow_{B_{KR}} y) \geq a.$$

Removing killing. The spatial forests can be coupled increasingly as the killing rate decreases. To see this, orient the edges arbitrarily and let $\nabla : \ell^2(E) \rightarrow \ell^2(V)$ send an edge basis vector to $\sqrt{c_e}(\delta_{e^+} - \delta_{e^-})$. The transfer-current formula gives the spatial-edge kernel

$$(4.10) \quad \mathcal{K}_m = \nabla^*(\mathcal{L} + mI)^{-1}\nabla, \quad \mathcal{L} = \nabla\nabla^*,$$

first in finite wired sets and then by exhaustion [16, Chapters 4 and 10]. For $0 < m \leq \mu$, resolvent order gives $0 \preceq \mathcal{K}_\mu \preceq \mathcal{K}_m \preceq I$. The determinantal domination theorem [15, Theorem 2.9] therefore couples the spatial edge sets with $\mathfrak{F}_\mu \subseteq \mathfrak{F}_m$. The event in (4.9) is increasing, so its lower bound holds for every $0 < m \leq \mu$.

On this event the common ancestor of x, y lies on their connecting path, hence in B_{KR} . Put

$$h_m(u) = \mathbb{P}_m(u \in \gamma_0^m), \quad A_m(t) = \sum_{u \in B_t} h_m(u).$$

Apply (1.4) with $f = \mathbf{1}_{B_R}$ to each singleton target in B_{KR} . Translation invariance gives

$$(4.11) \quad \begin{aligned} a|B_R|^2 &\leq \sum_{v \in B_{KR}} \sum_{x, y \in B_R} \mathbb{P}_m(\text{LCA}(x, y) = v) \\ &\leq \sum_{v \in B_{KR}} \left(\sum_{x \in B_R} h_m(v - x) \right)^2 \leq |B_{KR}| A_m((K+1)R)^2. \end{aligned}$$

Thus $A_m((K+1)R) \geq cR^{3/2}$, uniformly for $0 < m \leq R^{-2}$.

Couple the walks at different killing rates by stopping a single infinite walk at time E/m , with E an independent rate-one exponential variable. For a fixed vertex u , membership in the loop-erasure of a finite initial segment eventually stabilizes: after the last visit to u , that vertex can be erased at most once and cannot be reinserted. Its eventual membership is that in the infinite loop-erasure. Consequently $h_m(u) \rightarrow \mathbb{P}(u \in \eta)$ as $m \downarrow 0$. Take this limit in the finite sum $A_m((K+1)R)$ and rescale R by the fixed factor $K+1$. This proves (1.10); decreasing c includes bounded $R \geq 1$.

The growth exponent. For independent simple random walks S^1, S^2 started at 0, define

$$(4.12) \quad \text{Es}(r) = \mathbb{P}(\text{LE}(S^1[0, \tau_r^1]) \cap S^2[1, \tau_r^2] = \emptyset),$$

where τ_r^i is the first exit of S^i from B_r . Li–Shiraishi [13, Corollary 1.3] give

$$(4.13) \quad \text{Es}(r) \asymp r^{\beta-2}, \quad r \geq 1.$$

The finite-volume one-point estimate of Shiraishi [21, Proposition 8.2], followed by the membership limit above and the Green-function bound $\sum_{j \geq 0} \mathbb{P}_0(S_j = z) \leq C/(1 + |z|)$ [12, Theorem 4.3.1], gives

$$(4.14) \quad \mathbb{P}(z \in \eta) \leq \frac{C}{1 + |z|} \text{Es}(|z|), \quad z \neq 0.$$

Summing over lattice shells yields

$$(4.15) \quad \mathbb{E}|\eta \cap B_R| \leq CR^\beta.$$

Together with (1.10), this implies $\beta \geq 3/2$. □

5. FURTHER APPLICATIONS

5.1. Intersection of independent infinite LERWs. Lyons, Peres and Schramm [17, Question 6.3] asked whether loop-erasures of two independent copies of a transient Markov chain preserve a.s. infinite intersection. Theorem 1.3 gives the three-dimensional case.

Theorem 5.1. *Let η_a and η_b be independent infinite LERWs on $\mathbb{Z}^3$, started at arbitrary $a, b \in \mathbb{Z}^3$. Then*

$$\mathbb{P}(|\eta_a \cap \eta_b| = \infty) = 1.$$

Proof. Put $q = 3 - \beta$ and $\kappa = 3 - 2q = 2\beta - 3 \geq 0$. By (4.13) and (4.14), $\mathbb{P}(z \in \eta_x) \leq C(1 + |z - x|)^{-q}$. Write $B(x, n) = x + B_n$ and $|z|_\infty = \max_i |z_i|$. Let $e_1 = (1, 0, 0)$, and set

$$D_n(x) = \{z \in \mathbb{Z}^3 : |z - x - (n/2)e_1|_\infty < n/1000\}, \quad \sigma_n^x = \inf\{j : \eta_x(j) \notin B(x, n)\}.$$

Proposition 3.16 of [3], with $K = \emptyset$ and a translation of the entire configuration, gives, uniformly for distinct $z, w \in D_n(x)$,

$$(5.1) \quad \mathbb{P}(z \in \eta_x[0, \sigma_n^x]) \geq cn^{-q}, \quad \mathbb{P}(z, w \in \eta_x[0, \sigma_n^x]) \leq Cn^{-q}|z - w|^{-q}.$$

Choose an integer $n_0 \geq 10^5(1 + |a| + |b|)$ and put $n_j = 2^j n_0$. The sets $Q_j = D_{n_j}(a) \cap D_{n_j}(b)$ are disjoint, have cardinality comparable to n_j^3 , and lie at distance comparable to n_j from both starting points. Define

$$Z_j = |Q_j \cap \eta_a[0, \sigma_{n_j}^a] \cap \eta_b[0, \sigma_{n_j}^b]|.$$

Independence and (5.1) yield

$$(5.2) \quad \mathbb{E}Z_j \asymp n_j^\kappa, \quad \mathbb{E}Z_j^2 \leq Cn_j^\kappa \left(1 + \sum_{1 \leq |u| \leq n_j} |u|^{-2q}\right) \leq \begin{cases} Cn_j^{2\kappa}, & \kappa > 0, \\ C(1 + \log n_j), & \kappa = 0. \end{cases}$$

Here the one-point upper bound controls the diagonal term.

For $i < j$, $z \in Q_i$, $w \in Q_j$ and $x \in \{a, b\}$, [21, Proposition 8.2, case (I), with $k = 2$] gives

$$(5.3) \quad \mathbb{P}(z, w \in \eta_x) \leq Cn_i^{-q}n_j^{-q}.$$

Indeed, apply that estimate in $B(x, N)$ and let $N \rightarrow \infty$ with x, z, w fixed. Transience gives stabilization of loop-erasure at these vertices. The two permutation terms are bounded by $Cn_i^{-1} \text{Es}(n_i)n_j^{-1} \text{Es}(n_j)$ and $Cn_j^{-2} \text{Es}(n_j)^2$, respectively, using $|z - x| \asymp n_i$ and $|w - x| \asymp |z - w| \asymp n_j$. It follows by independence that $\mathbb{E}(Z_i Z_j) \leq Cn_i^\kappa n_j^\kappa$.

If $\kappa > 0$, the second-moment inequality applied to Z_j gives $\mathbb{P}(Z_j \geq cn_j^\kappa) \geq c_0 > 0$. If $\kappa = 0$, let $W_K = \sum_{j=1}^K Z_j$. The preceding bounds give

$$\mathbb{E}W_K \geq cK, \quad \mathbb{E}W_K^2 \leq C(K^2 + K \log n_0).$$

For $K \geq 1 + \log n_0$, the same inequality gives $\mathbb{P}(W_K \geq cK) \geq c_0$. Since the Q_j are disjoint, the event that the corresponding lower-bound events occur for infinitely many j forces infinitely many distinct common vertices. Moreover, $\mathbb{P}(\limsup_j E_j) \geq \limsup_j \mathbb{P}(E_j)$ for any events E_j ; in the critical case apply this to a sequence of K 's for which $W_K \geq cK$. Hence, in either case,

$$(5.4) \quad \mathbb{P}(|\eta_a \cap \eta_b| = \infty) \geq c_0,$$

where c_0 is independent of a, b .

Finally, a simple random walk on $\mathbb{Z}^3$ has infinitely many cut times almost surely [11, Theorem 1.2]. Deleting a finite initial segment therefore changes its loop-erasure only by finitely many vertices: choose a cut time beyond that segment, after which the two loop erasures have the same tail. Let $A = \{|\eta_a \cap \eta_b| = \infty\}$, and let $\mathcal{F}_n = \sigma(S_j^a, S_j^b : 0 \leq j \leq n)$ for the two underlying walks S^a, S^b . The Markov property and (5.4) imply

$$\mathbb{P}(A \mid \mathcal{F}_n) = \mathbb{P}(|\eta_{S_n^a} \cap \eta_{S_n^b}| = \infty) \geq c_0,$$

where the paths on the right are fresh independent LERWs. Martingale convergence now gives $\mathbf{1}_A \geq c_0$ almost surely. $\square$

5.2. LERW length and forest volume under uniform killing. The separation inequality also converts random-walk intersection bounds into lower bounds on LERW length under uniform killing. Let G be a finite transitive network, or a countable locally finite network with a transitive conductance-preserving group Γ satisfying

$$(5.5) \quad \sum_y F(o, y) = \sum_x F(x, o)$$

for every nonnegative diagonally Γ -invariant function F on $V \times V$. Finite transitive networks satisfy (5.5); on $\mathbb{Z}^d$, translations give it by a change of variables. Normalize the total jump rate to 1, and use the massive spanning forests in Section 3, with its infinite-volume construction from Section 4. Recall the definition of γ_o^m in Section 3. Set

$$\chi(m) = \mathbb{E}|C_m(o)|, \quad L(m) = \mathbb{E}|\gamma_o^m|,$$

and retain g_m and $\mathcal{B}(m)$ from (4.1).

Proposition 5.2. *For every $m > 0$,*

$$(5.6) \quad \frac{1}{2^{10}m^2\mathcal{B}(m)} \leq \chi(m) \leq L(m)^2, \quad \frac{1}{32m\sqrt{\mathcal{B}(m)}} \leq L(m) \leq 1 + m^{-1}.$$

Proof. We first prove $\chi(m) \leq L(m)^2$. For each spatial vertex v and each finite set $F \subseteq V$, apply (1.4) with $B = \{v\}$ and $f = \mathbf{1}_F$. Letting $F \uparrow V$ and using monotone convergence gives

$$A(v) := \sum_{x,y \in V} \mathbb{P}_m(\text{LCA}(x, y) = v) \leq \left(\sum_{x \in V} \mathbb{P}_m(v \in \gamma_x^m) \right)^2.$$

Apply (5.5) to $F(x, v) = \mathbb{P}_m(v \in \gamma_x^m)$ and to $F(x, v) = \sum_y \mathbb{P}_m(\text{LCA}(x, y) = v)$. They give

$$\sum_x \mathbb{P}_m(o \in \gamma_x^m) = L(m), \quad A(o) = \sum_{v,y} \mathbb{P}_m(\text{LCA}(o, y) = v) = \chi(m).$$

Tonelli's theorem permits these identities before finiteness of $\chi(m)$ is known. It follows that $\chi(m) \leq L(m)^2$. The expected number of jumps before killing is m^{-1} , so $L(m) \leq 1 + m^{-1}$.

For the lower bound, write $I_y = I_{o,y}$ in (4.3). Sum (4.4)–(4.5) over y , using (4.2) and transitivity:

$$(5.7) \quad \sum_y \mathbb{E}I_y = m^{-2}, \quad \sum_y \mathbb{E}I_y^2 = 4m^{-2}\mathcal{B}(m).$$

Cauchy–Schwarz, first on finite sets and then by exhaustion, gives

$$\sum_y \mathbb{P}(I_y > 0) \geq \frac{(\sum_y \mathbb{E}I_y)^2}{\sum_y \mathbb{E}I_y^2} = \frac{1}{4m^2\mathcal{B}(m)}.$$

Summing (4.7) yields the lower bound on $\chi(m)$. The lower bound on $L(m)$ follows from $\chi(m) \leq L(m)^2$. $\square$

Proposition 5.2 has the following consequence on the long-range LERW on $\mathbb{Z}^3$.

Proposition 5.3. *Let S be a symmetric translation-invariant random walk on $\mathbb{Z}^3$, started at 0, with jump distribution $q(z) \asymp |z|^{-3-\alpha}$, $3/2 < \alpha < 2$, and let τ_R be its first exit from B_R . Then*

$$(5.8) \quad \beta_\alpha := \liminf_{R \rightarrow \infty} \frac{\log \mathbb{E}|\text{LE}(S[0, \tau_R])|}{\log R} \in \left[\frac{3}{2}, \alpha \right].$$

Furthermore, $\beta_\alpha \rightarrow 3/2$ as $\alpha \downarrow 3/2$.

Proof. Indeed, replace jumps longer than r by killing at rate $m_r = \sum_{|z|>r} q(z) \asymp r^{-\alpha}$. Then the corresponding $\mathcal{B}(m_r)$ is bounded by a constant multiple of that for S . Proposition 5.2 and the heat-kernel estimates [6, Theorem 1.1] give expected loop-erased length at least $O(r^{3/2})$. Restricting to trajectories contained in $B_{Ar \log r}$, for sufficiently large fixed A , preserves this

lower bound. Comparing their probabilities with those of S stopped at exit loses only powers of $\log r$, proving (5.8). On the other hand, note that $\mathbb{E}|\text{LE}(S[0, \tau_R])| \leq 1 + \mathbb{E}\tau_R \leq CR^\alpha$. Hence, we have $3/2 \leq \beta_\alpha \leq \alpha$, and $\beta_\alpha \rightarrow 3/2$ as $\alpha \downarrow 3/2$. $\square$

We refer the reader to Xiao–Pan–Fan–Deng [23] for numerical estimates of β_α .

5.3. Wired spanning forests and square summability. The next result compares wired forest connectivity with the overlap of independent LERWs. Let $\mathcal{G} = (V, E)$ be a connected, locally finite transient network. Write $\mathfrak{F}$ for its wired uniform spanning forest, and set

$$h_x(v) = \mathbb{P}(v \in \eta_x), \quad \tau(x, y) = \mathbb{P}(x \leftrightarrow y \text{ in } \mathfrak{F}), \quad G(x, v) = \sum_{n \geq 0} \mathbb{P}_x(S_n = v),$$

where η_x is infinite loop-erasure and S has transition kernel $p(x, y) = c_{xy}/c(x)$, with $c(x) = \sum_y c_{xy}$.

Proposition 5.4. *For every $x, y \in V$,*

$$(5.9) \quad \tau(x, y) \leq 2 \sum_{v \in V} h_x(v) h_y(v).$$

If $\mathcal{G}$ is transitive, then for any $o \in V$,

$$(5.10) \quad \sum_{v \in V} h_o(v)^2 < \infty \iff \sum_{v \in V} G(o, v)^2 < \infty.$$

On a transitive network, two independent infinite LERWs from o therefore have a finite expected number of common vertices exactly when the Green function is square summable.

Proof. Use Wilson’s algorithm rooted at infinity [16, Chapter 10], with each edge oriented along its branch. Every vertex has one outgoing edge. The forward rays of adjacent vertices coalesce, hence so do the rays of any two vertices in the same component. Their first common vertex $\text{LCA}(x, y)$ is therefore defined exactly on $\{x \leftrightarrow y\}$.

Take a finite connected exhaustion (V_N) , with its complement wired to a root. Couple the finite and infinite algorithms using independent walks X, Y from x, y , processed in this order. Let η_x^N be the loop-erasure of X up to exiting V_N , with the wired root omitted. By transience, η_x^N eventually agrees with η_x on every fixed finite vertex set. If Y first hits η_x at time $T < \infty$, its first hit of η_x^N is also at T for all sufficiently large N . If $T = \infty$, its finite-volume attachment is eventually different from any prescribed vertex v : Y visits v only finitely often, and membership in η_x^N stabilizes on those finite prefixes. Consequently, the singleton ancestor probabilities and the one-point functions converge. Applying (1.5) with $B = \{v\}$ gives

$$\mathbb{P}(\text{LCA}(x, y) = v) \leq 2h_x(v)h_y(v).$$

Summing over v proves (5.9).

For (5.10), assume transitivity. Since $h_o(v) \leq \mathbb{P}_o(S \text{ hits } v) = G(o, v)/G(o, o)$, square summability of $G(o, \cdot)$ implies that of h_o . Conversely, suppose $h_o \in \ell^2(V)$. Transitivity, truncation to finite graph balls, and Cauchy–Schwarz give

$$\sum_v h_o(v) h_y(v) \rightarrow 0 \quad \text{as } \text{dist}(o, y) \rightarrow \infty.$$

Indeed, the truncations supported on $B(o, R)$ and $B(y, R)$ are disjoint when $\text{dist}(o, y) > 2R$, and both remainder norms equal $\|h_o \mathbf{1}_{B(o, R)^c}\|_2$. Thus (5.9) implies $\tau(o, y) \rightarrow 0$. If the Green-function square sum were infinite, [17, Theorem 1.4] would give infinitely many intersections of two independent walks from o almost surely. The same holds from arbitrary starting vertices, by forcing finite positive-probability paths from o to those vertices and discarding these finite prefixes. By [17, Theorem 1.1], the second walk then hits the infinite loop-erasure of the first almost surely. Wilson’s algorithm gives $\tau(o, y) = 1$ for every y , a contradiction. Finally, independence and Tonelli’s theorem identify $\sum_v h_o(v)^2$ with the expected overlap of two independent copies. $\square$

REFERENCES

- [1] N. Alon, A. Nachmias, and M. Shalev, *The diameter of the uniform spanning tree of dense graphs*, *Combin. Probab. Comput.* **31** (2022), no. 6, 1010–1030. doi:10.1017/S0963548322000062.
- [2] N. Anari, K. Liu, S. Oveis Gharan, and C. Vintzant, *Log-concave polynomials III: Mason’s ultra-log-concavity conjecture for independent sets of matroids*, *Proc. Amer. Math. Soc.* **152** (2024), no. 5, 1969–1981. doi:10.1090/proc/16724.
- [3] O. Angel, D. A. Croydon, S. Hernández-Torres, and D. Shiraishi, *Scaling limits of the three-dimensional uniform spanning tree and associated random walk*, *Ann. Probab.* **49** (2021), no. 6, 3032–3105. doi:10.1214/21-AOP1523.
- [4] E. Archer, A. Nachmias, and M. Shalev, *The GHP scaling limit of uniform spanning trees in high dimensions*, *Comm. Math. Phys.* **405** (2024), article no. 73. doi:10.1007/s00220-023-04923-2.
- [5] E. Archer and M. Shalev, *The GHP scaling limit of uniform spanning trees of dense graphs*, *Random Structures Algorithms* **65** (2024), no. 1, 149–190. doi:10.1002/rsa.21213.
- [6] R. F. Bass and D. A. Levin, *Transition probabilities for symmetric jump processes*, *Trans. Amer. Math. Soc.* **354** (2002), no. 7, 2933–2953.
- [7] I. Benjamini and G. Kozma, *Loop-erased random walk on a torus in dimensions 4 and above*, *Comm. Math. Phys.* **259** (2005), no. 2, 257–286. doi:10.1007/s00220-005-1388-4.
- [8] P. Brändén and J. Huh, *Lorentzian polynomials*, *Ann. of Math. (2)* **192** (2020), 821–891. doi:10.4007/annals.2020.192.3.4.
- [9] F. Chung, P. Horn, and L. Lu, *Diameter of random spanning trees in a given graph*, *J. Graph Theory* **69** (2012), no. 3, 223–240. doi:10.1002/jgt.20577.
- [10] G. F. Lawler, *A lower bound on the growth exponent for loop-erased random walk in two dimensions*, *ESAIM Probab. Stat.* **3** (1999), 1–21.
- [11] G. F. Lawler, *Cut times for simple random walk*, *Electron. J. Probab.* **1** (1996), paper no. 13, 1–24. doi:10.1214/EJP.v1-13.
- [12] G. F. Lawler and V. Limic, *Random Walk: A Modern Introduction*, *Cambridge Studies in Advanced Mathematics*, vol. 123, Cambridge University Press, 2010.
- [13] X. Li and D. Shiraishi, *One-point function estimates for loop-erased random walk in three dimensions*, *Electron. J. Probab.* **24** (2019), paper no. 111, 46 pp. doi:10.1214/19-EJP361.
- [14] R. Liu, *An explicit lower bound for the growth exponent of three-dimensional loop-erased random walk*, *arXiv:2609.18643v1* (2026).
- [15] R. Lyons, *Determinantal probability: basic properties and conjectures*, in *Proceedings of the International Congress of Mathematicians—Seoul 2014*, vol. IV, 137–161; arXiv:1406.2707.
- [16] R. Lyons and Y. Peres, *Probability on Trees and Networks*, *Cambridge Series in Statistical and Probabilistic Mathematics*, vol. 42, Cambridge University Press, 2016.
- [17] R. Lyons, Y. Peres, and O. Schramm, *Markov chain intersections and the loop-erased walk*, *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), 779–791.
- [18] P. Michaeli, A. Nachmias, and M. Shalev, *The diameter of uniform spanning trees in high dimensions*, *Probab. Theory Related Fields* **179** (2021), 261–294.
- [19] Y. Peres and D. Revell, *Scaling limits of the uniform spanning tree and loop-erased random walk on finite graphs*, *arXiv:math/0410430v2* (2005).
- [20] J. Schweinsberg, *The loop-erased random walk and the uniform spanning tree on the four-dimensional discrete torus*, *Probab. Theory Related Fields* **144** (2009), 319–370. doi:10.1007/s00440-008-0149-7.
- [21] D. Shiraishi, *Growth exponent for loop-erased random walk in three dimensions*, *Ann. Probab.* **46** (2018), 687–774.
- [22] D. Shiraishi, *Hausdorff dimension of the scaling limit of loop-erased random walk in three dimensions*, *Ann. Inst. H. Poincaré Probab. Statist.* **55** (2019), 791–834.
- [23] T. Xiao, X. Pan, Z. Fan, and Y. Deng, *Scaling of long-range loop-erased random walks*, *arXiv:2603.27992v2* (2026).